

Analysis of Algorithms

Data Structures and Algorithms for Computational Linguistics III
(ISCL-BA-07)

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What are we analyzing?

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- So far, we frequently asked: ‘can we do better?’
- Now, we turn to the questions of
 - what is better?
 - how do we know an algorithm is better than the other?
- There are many properties that we may want to improve
 - correctness
 - robustness
 - simplicity
 - ...
 - In this lecture, *efficiency* will be our focus
 - in particular time efficiency/complexity

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write the code, experiment

- A possible approach:
 - Implement the algorithm
 - Test with varying input
 - Analyze the results

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 - Implementing something that does not work is not productive (or fun)
 - It is often not possible cover all potential inputs
 - If your version takes 10 seconds less than a version reported 10 years ago, do you really have an improvement?

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- A possible approach:
 - Implement the algorithm
 - Test with varying input
 - Analyze the results
- A formal approach offers some help here
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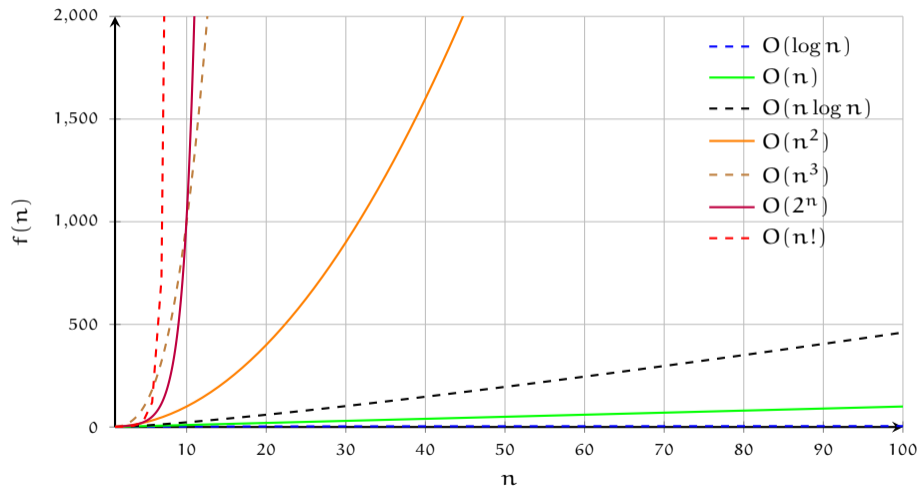
Some functions to know about

Family	Definition
Constant	$f(n) = c$
Logarithmic	$f(n) = \log_b n$
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- We will use these functions to characterize running times of algorithms

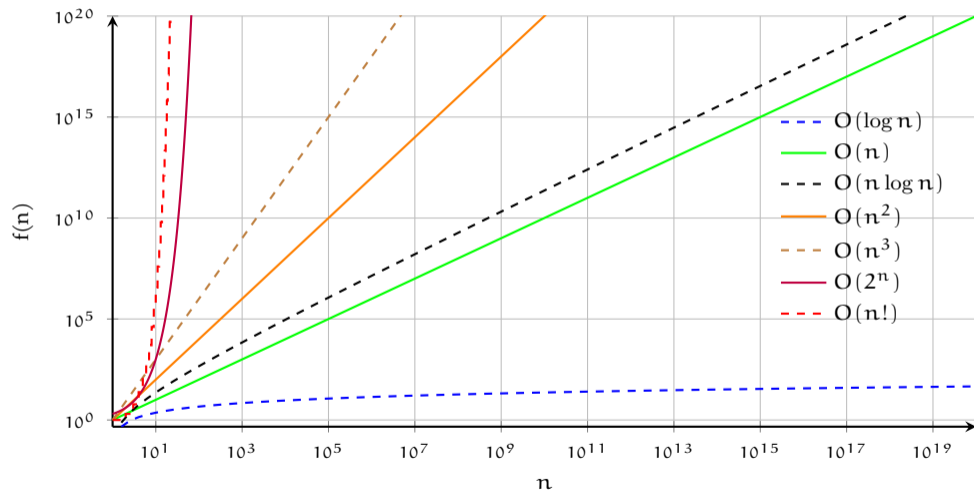
Some functions to know about

the picture - why we care about their difference



Some functions to know about

the bigger picture



A few facts about logarithms

- Logarithm is the inverse of exponentiation:

$$x = \log_b n \iff b^x = n$$

- We will mostly use base-2 logarithms. For us, no-base means base-2
- Additional properties:

$$\log xy = \log x + \log y$$

$$\log \frac{x}{y} = \log x - \log y$$

$$\log x^a = a \log x$$

$$\log_b x = \frac{\log_k x}{\log_k b}$$

- Logarithmic functions grow (much) slower than linear functions

Polynomials

- A degree-0 polynomial is a constant function ($f(n) = c$)
- Degree-1 is linear ($f(n) = n + c$)
- Degree-2 is quadratic ($f(n) = n^2 + n + c$)
- ...
- We generally drop the lower order terms (soon we'll explain why)
- Sometimes it will be useful to remember that

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

Combinations and permutations

- $n! = n \times (n - 1) \times \dots \times 2 \times 1$
- Permutations:

$$P(n, k) = n \times (n - 1) \times \dots \times (n - k + 1) = \frac{n!}{(n - k)!}$$

- Combinations 'n choose k':

$$C(n, k) = \binom{n}{k} = \frac{P(n, k)}{P(k, k)} = \frac{n!}{(n - k)! \times k!}$$

Proof by induction

- Induction is an important proof technique
- It is often used for both proving the correctness and running times of algorithms
- It works if we can enumerate the steps of an algorithm (loops, recursion)
 - Show that base case holds
 - Assume the result is correct for n , show that it also holds for $n + 1$

Proof by induction

Example: show that $1 + 2 + 3 + \dots + n = n(n + 1)/2$

- Base case, for $n=1$

$$(1 \times 2)/2 = 1$$

- Assuming

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

we need to show that

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

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$$\frac{n(n+1)}{2} + (n+1)$$

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$$\frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

Formal analysis of running time of algorithms

- We are focusing on characterizing running time of algorithms
- The running time is characterized as a function of input size
- We are aiming for an analysis method
 - independent of hardware / software environment
 - does not require implementation before analysis
 - considers all possible inputs

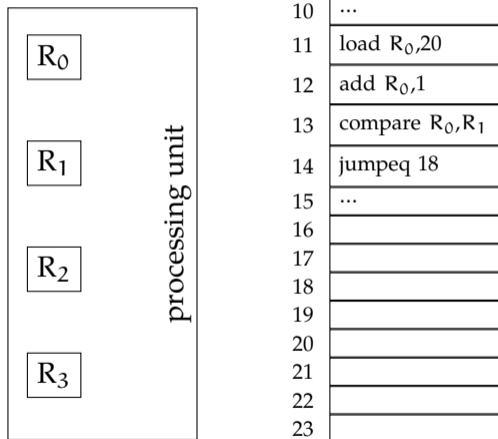
How much hardware independence?

How much hardware independence?

quite, but not completely: we assume a RAM model of computing

- Characterized by random access memory (RAM) (e.g., in comparison to a sequential memory, like a tape)
- We assume the system can perform some primitive operations (addition, comparison) in constant time
- The data and the instructions are stored in the RAM
- The processor fetches them as needed, and executes following the instructions
- This is largely true for any computing system we use in practice

RAM model: an example



- Processing unit performs basic operations in constant time
- Any memory cell with an address can be accessed in equal (constant) time
- The instructions as well as the data is kept in the memory
- There may be other, specialized registers
- Modern processing units also employ a 'cache'

Formal analysis of running time

- Simply count the number of primitive operations
- Primitive operations include:
 - Assignment
 - Arithmetic operations
 - Comparing primitive data types (e.g., numbers)
 - Accessing a single memory location
 - Function calls, return from functions
- **Not** primitive operations:
 - loops, recursion
 - comparing sequences

Focus on the worst case

- Algorithms are generally faster on certain input than others
- In most cases, we are interested in the *worst case* analysis
 - Guaranteeing worst case is important
 - It is also relatively easier: we need to identify the worst-case input
- Average case analysis is also useful, but
 - requires defining a distribution over possible inputs
 - often more challenging

Counting primitive operations

example: nearest points, the naive algorithm

```
def shortest_distance(points):
    n = len(points)           # 2 (constant?)
    min = 0                   # 1 (constant)
    for i in range(n):       # n times
        for j in range(i):   # i times
            d = distance(points[i], points[j]) # 2? (constant)
            if min > d:      # 1 (constant)
                min = d     # 1 (constant)
    return min                # 1 (constant)
```

$$\begin{aligned}
 T(n) &= 3 + (1 + 2 + 3 + \dots + n - 1) \times 4 + 1 \\
 &= 4 \times \frac{(n-1)(n-2)}{2} + 4
 \end{aligned}$$

Big-O notation

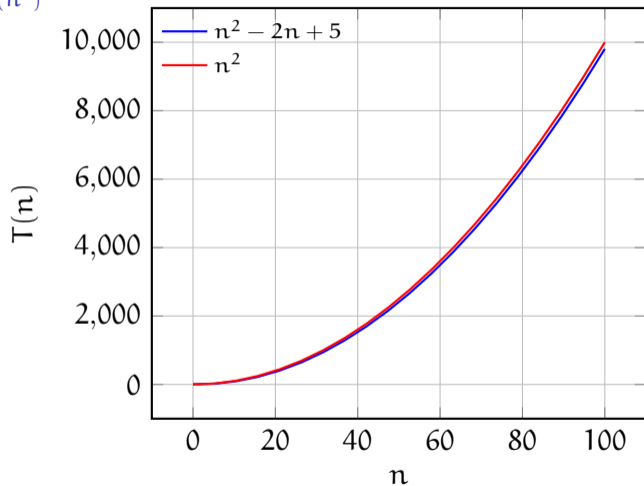
- Big-O notation is used for indicating an upper bound on running time of an algorithm as a function of running time
- If running time of an algorithm is $O(f(n))$, its running time grows proportional to $f(n)$ as the input size n grows
- More formally, given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there is a constant $c > 0$ and integer $n_0 \geq 1$ such that

$$f(n) \leq c \times g(n) \text{ for } n \geq n_0$$

- Sometimes the notation $f(n) = O(g(n))$ is also used, but beware: this equal sign is not symmetric

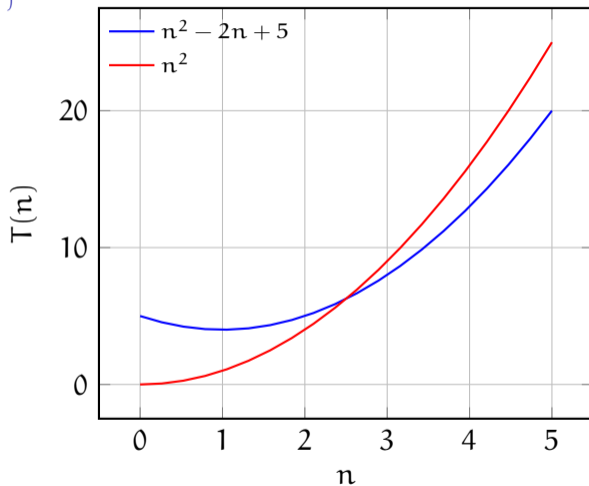
Big-O example

$T(n) = n^2 - 2n + 5$ is $O(n^2)$



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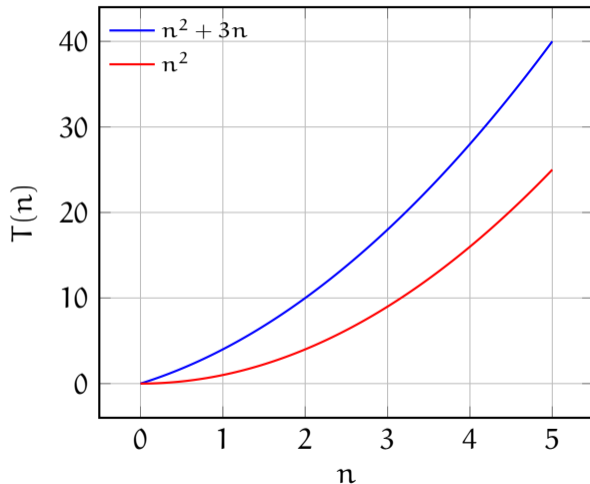
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Not surprising: $T(n) < n^2$ for $n \geq 3$

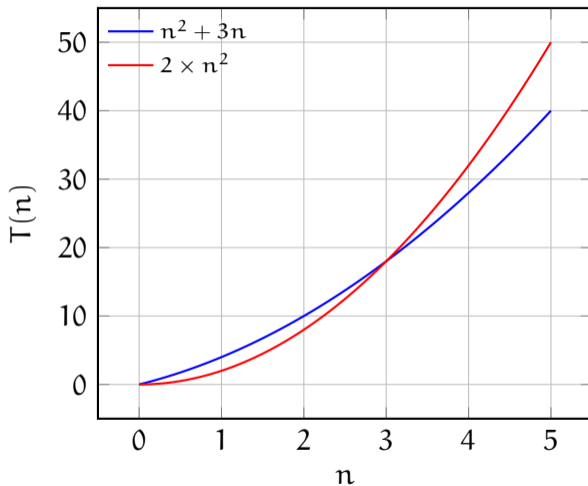
Big-O, another example

$T(n) = n^2 + 3n$ is $O(n^2)$



Big-O, another example

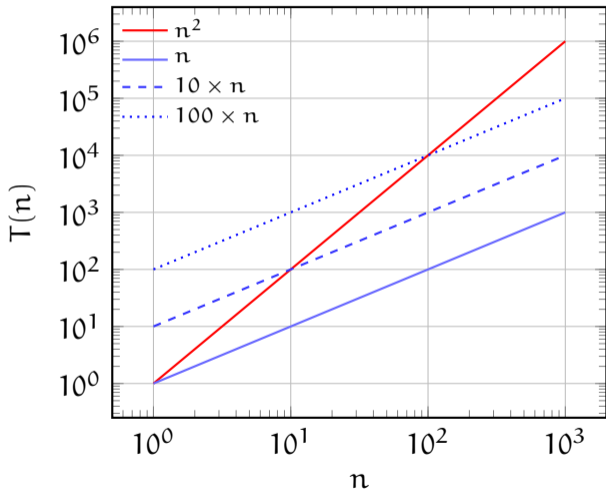
$T(n) = n^2 + 3n$ is $O(n^2)$



$$T(n) < 2 \times n^2 \text{ for } n \geq 4$$

Big-O, yet another example

but n^2 is not $O(n)$ – proof by picture



Back to the function classes

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- None of these functions can be expressed as a constant factor of another

Rules of thumb

Drop the lower order terms

- In the big-O notation, we drop the constants and lower order terms
 - Any polynomial degree d is $O(n^d)$
 $10n^3 + 4n^2 + n + 100$ is $O(n^3)$
 - Drop any lower order terms:
 $2^n + 10n^3$ is $O(2^n)$
- Use the simplest expression:
 - $5n + 100$ is $O(5n)$, but we prefer $O(n)$
 - $4n^2 + n + 100$ is $O(n^3)$,
- Transitivity: if $f(n) = O(g(n))$, and $g(n) = O(h(n))$, then $f(n) = O(h(n))$
- Additivity: if both $f(n)$ and $g(n)$ are $O(h(n))$ $f(n) + g(n)$ is $O(h(n))$

Rules of thumb

examples

$$\frac{f(n) \quad O(f(n))}{7n - 2}$$

Rules of thumb

examples

$f(n)$	$O(f(n))$
$7n - 2$	n
$3n^3 - 2n^2 + 5$	

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$n2^n$	$n2^n$
$\log n!$	$n \log n$

Big-O: back to nearest points

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 &= O(n^2)
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Big-O examples

linear search

- What is the worst-case running time?

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1 def linear_search(seq, val):
2     i, n = 0, len(seq)
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4         if seq[i] == val:
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7     return None
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 3. $2n$ comparisons, n increment
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Note: do not confuse the big-O with the worst case analysis.

Recursive example

Recursive binary search

```

1 def rbs(a, x, L=0, R=n):
2     if L >= R:
3         return None
4     M = (L + R) // 2
5     if a[M] == x:
6         return M
7     if a[M] > x:
8         return rbs(a, x, L,
9             ↪ M - 1)
10    else:
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- So, $T(n) = 2c + T(n/4) = 3c + T(n/8)$
- More generally, $T(n) = ic + T(n/2^i)$
- Recursion terminates when $n/2^i = 1$, or $n = 2^i$,
the good news: $i = \log n$

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- So, $T(n) = 2c + T(n/4) = 3c + T(n/8)$
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Recursive example

Recursive binary search

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1 def rbs(a, x, L=0, R=n):
2     if L >= R:
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4     M = (L + R) // 2
5     if a[M] == x:
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7     if a[M] > x:
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You do not always need to prove: for most recurrence relations, there is a way to obtain quick solutions (we are not going to cover it further, see Appendix)

Why asymptotic analysis is important?

'maximum problem size'

- Assume we can solve a problem of size m in a given time on current hardware
- We get a better computer, which runs 1024 times faster
- New problem size we can solve in the same time

Complexity	new problem size
Linear (n)	$1024m$
Quadratic (n^2)	$32m$
Exponential (2^n)	$m + 10$

- This also demonstrates the gap between polynomial and exponential algorithms:
 - with a exponential algorithm fast hardware does not help
 - problem size for exponential algorithms does not scale with faster computers

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pros and cons

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 - A constant factor of 100^{100} should probably not be ignored

Big-O relatives

- Big-O (upper bound): $f(n)$ is $O(g(n))$
if $f(n)$ is asymptotically *less than or equal to* $g(n)$

$$f(n) \leq cg(n) \text{ for } n > n_0$$

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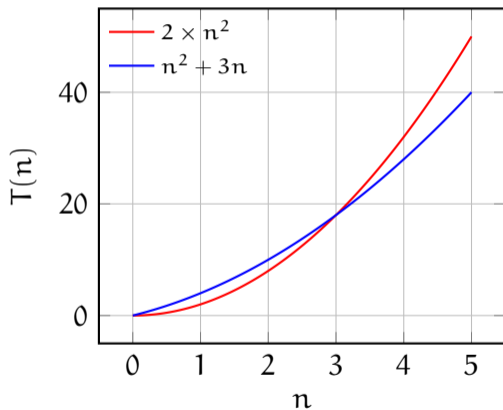
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- Big-Theta (upper/lower bound): $f(n)$ is $\Theta(g(n))$
if $f(n)$ is asymptotically *equal to* $g(n)$

$$f(n) \text{ is } O(g(n)) \text{ and } f(n) \text{ is } \Omega(g(n))$$

Big-O, Big-Ω, Big-Θ: an example

$T(n) = n^2 + 3n$ is $O(n^2)$

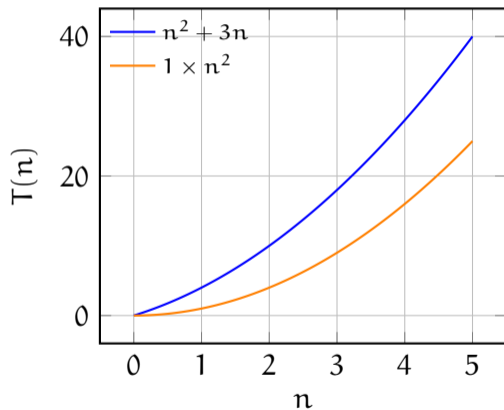


O for $c = 2$ and $n_0 = 3$

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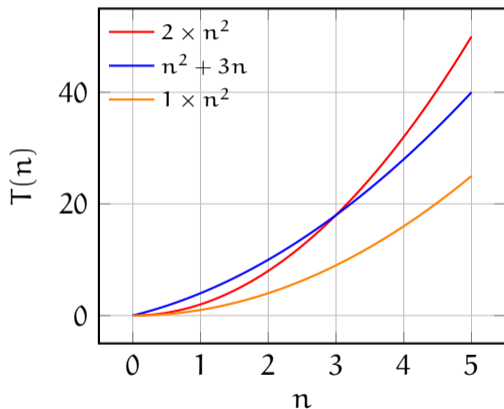
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Ω for $c = 1$ and $n_0 = 0$

$$T(n) \geq c g(n) \text{ for } n > n_0$$

Big-O, Big- Ω , Big- Θ : an example

$T(n) = n^2 + 3n$ is $\Theta(n^2)$



O for $c = 2$ and $n_0 = 3$

$$T(n) \leq c g(n) \text{ for } n > n_0$$

Ω for $c = 1$ and $n_0 = 0$

$$T(n) \geq c g(n) \text{ for } n > n_0$$

Θ for $c = 2$, $n_0 = 3$, $c' = 1$ and $n'_1 = 0$

$$T(n) \leq c g(n) \text{ for } n > n_0 \quad \text{and}$$

$$T(n) \geq c' g(n) \text{ for } n > n'_0$$

Summary

- Algorithmic analysis mainly focuses on worst-case asymptotic running times
- *Sublinear* (e.g., *logarithmic*), *Linear* and $n \log n$ algorithms are good
- *Polynomial* algorithms may be acceptable in many cases
- *Exponential* algorithms are bad
- We will return to concepts from this lecture while studying various algorithms
- Reading for this lectures: Goodrich, Tamassia, and Goldwasser (2013, chapter 3)

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Next:

- Common patterns in algorithms
- Sorting algorithms
- Reading: Goodrich, Tamassia, and Goldwasser (2013, chapter 12) – up to 12.7

Acknowledgments, credits, references

- Some of the slides are based on the previous year's course by Corina Dima.



Goodrich, Michael T., Roberto Tamassia, and Michael H. Goldwasser (2013). *Data Structures and Algorithms in Python*. John Wiley & Sons, Incorporated. ISBN: 9781118476734.

A(nother) view of computational complexity

P, NP, NP-complete and all that

- A major division of complexity classes according to Big-O notation is between
 - P polynomial time algorithms
 - NP non-deterministic polynomial time algorithms
- A big question in computing is whether $P = NP$
- All problems in NP can be reduced in polynomial time to a problem in a subclass of NP (*NP-complete*)
 - Solving an NP complete problem in P would mean proving

$$P = NP$$

Video from <https://www.youtube.com/watch?v=YX40hbAHx3s>

Exercise

Sort the functions based on asymptotic order of growth

$$\log n^{1000}$$

$$n \log(n)$$

$$5^n$$

$$\log n$$

$$\log n^{1/\log n}$$

$$\log n$$

$$\log 2^n/n$$

$$\log n!$$

$$\log 2^n$$

$$\log 5^n$$

$$\binom{n}{n/2}$$

$$\log \log n!$$

$$\sqrt{n}$$

$$n^2$$

$$2^n$$

$$\binom{n}{2}$$

Recurrence relations

the master theorem

- Given a recurrence relation:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

a number of sub-problems

b reduction factor or the input

f(n) amount of work for creating and combining sub-problems

$$T(n) = \begin{cases} \Theta(n^{\log_b a}) & \text{if } f(n) \text{ is } O(n^{\log_b a - \epsilon}) \\ \Theta(n^{\log_b a} \log n) & \text{if } f(n) \text{ is } \Theta(n^{\log_b a}) \\ \Theta(f(n)) & \text{if } f(n) \text{ is } \Omega(n^{\log_b a + \epsilon}) \text{ and } af(n/b) \leq cf(n) \text{ for some } c < 1 \end{cases}$$

- In many practical cases $a = b$ (simplifies the expressions above)
- But the theorem is not general for all recurrences: it requires equal splits

