

FSA and regular languages

Data Structures and Algorithms for Computational Linguistics III
(ISCL-BA-07)

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Languages and automata

- Recognizing strings from a language defined by a grammar is a fundamental question in computer science
- The efficiency of computation, and required properties of computing device depends on the grammar (and the language)
- A well-known hierarchy of grammars both in computer science and linguistics is the *Chomsky hierarchy*
- Each grammar in the Chomsky hierarchy corresponds to an abstract computing device (an automaton)
- The class of *regular grammars* are the class that corresponds to *finite state automata*

How to describe a language?

Formal grammars

A formal *grammar* is a finite specification of a (formal) language.

- Since we consider languages as sets of strings, for a finite language, we can (conceivably) list all strings
- How to define an infinite language?
- Is the definition $\{ba, baa, baaa, baaaa, \dots\}$ 'formal enough'?
- Using regular expressions, we can define it as baa^*
- But we will introduce a more general method for defining languages

Phrase structure grammars

- A phrase structure grammar is a generative device
- If a given string can be generated by the grammar, the string is in the language
- The grammar generates *all* and the *only* strings that are valid in the language
- A phrase structure grammar has the following components
 - Σ A set of *terminal* symbols
 - N A set of *non-terminal* symbols
 - $S \in N$ A special non-terminal, called the start symbol
 - R A set of *rewrite rules* or *production rules* of the form:

$$\alpha \rightarrow \beta$$

which means that the sequence α can be rewritten as β (both α and β are sequences of terminal and non-terminal symbols)

Chomsky hierarchy and automata

<i>Grammar class</i>	<i>Rules</i>	<i>Automata</i>				
Unrestricted grammars	$\alpha \rightarrow \beta$	Turing machines				
Context-sensitive grammars	$\alpha A \beta \rightarrow \alpha \gamma \beta$	Linear-bounded automata				
Context-free grammars	$A \rightarrow \alpha$	Pushdown automata				
Regular grammars	<table border="1"> <tr> <td>$A \rightarrow a$</td> <td>$A \rightarrow a$</td> </tr> <tr> <td>$A \rightarrow aB$</td> <td>$A \rightarrow B a$</td> </tr> </table>	$A \rightarrow a$	$A \rightarrow a$	$A \rightarrow aB$	$A \rightarrow B a$	Finite state automata
$A \rightarrow a$	$A \rightarrow a$					
$A \rightarrow aB$	$A \rightarrow B a$					

Regular grammars: definition

A regular grammar is a tuple $G = (\Sigma, N, S, R)$ where

Σ is an alphabet of terminal symbols

N are a set of non-terminal symbols

S is a special 'start' symbol $\in N$

R is a set of rewrite rules following one of the following patterns ($A, B \in N$, $a \in \Sigma$, ϵ is the empty string)

Left regular

1. $A \rightarrow a$
2. $A \rightarrow Ba$
3. $A \rightarrow \epsilon$

Right regular

1. $A \rightarrow a$
2. $A \rightarrow aB$
3. $A \rightarrow \epsilon$

Regular languages: some properties/operations

$\mathcal{L}_1 \mathcal{L}_2$ Concatenation of two languages \mathcal{L}_1 and \mathcal{L}_2 : any sentence of \mathcal{L}_1 followed by any sentence of \mathcal{L}_2

\mathcal{L}^* Kleene star of \mathcal{L} : \mathcal{L} concatenated with itself 0 or more times

\mathcal{L}^R Reverse of \mathcal{L} : reverse of any string in \mathcal{L}

$\overline{\mathcal{L}}$ Complement of \mathcal{L} : all strings in $\Sigma_{\mathcal{L}}^*$ except the ones in \mathcal{L} ($\Sigma_{\mathcal{L}}^* - \mathcal{L}$)

$\mathcal{L}_1 \cup \mathcal{L}_2$ Union of languages \mathcal{L}_1 and \mathcal{L}_2 : strings that are in any of the languages

$\mathcal{L}_1 \cap \mathcal{L}_2$ Intersection of languages \mathcal{L}_1 and \mathcal{L}_2 : strings that are in both languages

Regular languages are closed under all of these operations.

Three ways to define a regular language

- A language is regular if there is regular grammar that generates/recognizes it
- A language is regular if there is an FSA that generates/recognizes it
- A language is regular if we can define a regular expressions for the language

Regular expressions

- Every regular language (RL) can be expressed by a regular expression (RE), and every RE defines a RL
- A RE e defines a RL $\mathcal{L}(e)$
- Relations between RE and RL
 - $\mathcal{L}(\emptyset) = \emptyset$,
 - $\mathcal{L}(\epsilon) = \epsilon$,
 - $\mathcal{L}(a) = a$
 - $\mathcal{L}(ab) = \mathcal{L}(a)\mathcal{L}(b)$
 - $\mathcal{L}(a^*) = \mathcal{L}(a)^*$
 - $\mathcal{L}(a|b) = \mathcal{L}(a) \cup \mathcal{L}(b)$
(some author use the notation $a+b$, we will use $a|b$ as in many practical implementations)

where, $a, b \in \Sigma$, ϵ is empty string, \emptyset is the language that accepts nothing (e.g., $\Sigma^* - \Sigma^*$)

- Note: no standard complement and intersection in RE

Regular expressions

and some extensions

- Kleene star (a^*), concatenation (ab) and union ($a|b$) are the basic operations
- Parentheses can be used to group the sub-expressions. Otherwise, the priority of the operators are as listed above: $a|bc^* = a|(b(c^*))$
- In practice some short-hand notations are common
 - $.$ = $(a_1 | \dots | a_n)$,
for $\Sigma = \{a_1, \dots, a_n\}$
 - $a^+ = aa^*$
 - $[a-c] = (a|b|c)$
 - $[\hat{a}-c] = . - (a|b|c)$
 - $\backslash d = (0|1|\dots|8|9)$
 - ...
- And some non-regular extensions, like $(a^*)b\backslash 1$ (sometimes the term *regex* is used for expressions with non-regular extensions)

Some properties of regular expressions

Useful identities for simplifying regular expressions

- $u|(v|w) = (u|v)|w$
- $u|v = v|u$
- $u(v|w) = uv|uw$
- $u|\emptyset = u$
- $u\epsilon = \epsilon u = u$
- $\emptyset u = \emptyset$
- $u(vw) = (uv)w$
- $\emptyset^* = \epsilon$
- $\epsilon^* = \epsilon$
- $(u^*)^* = u^*$
- $u|u = u$
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An exercise

Simplify $a|ab^*$

Note: some of these are direct statements of Kleene algebra, others can be derived from them.

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$$a|ab^* = a\epsilon|ab^*$$

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Simplify $a|ab^*$

$$\begin{aligned} a|ab^* &= a\epsilon|ab^* \\ &= a(\epsilon|b^*) \end{aligned}$$

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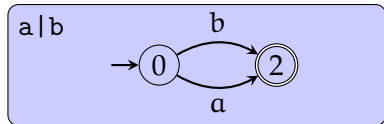
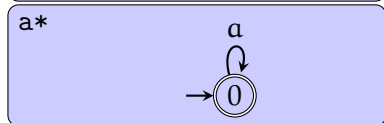
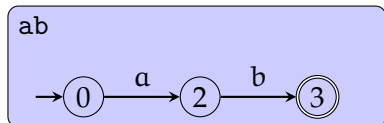
An exercise

Simplify $a|ab^*$

$$\begin{aligned} a|ab^* &= a\epsilon|ab^* \\ &= a(\epsilon|b^*) \\ &= ab^* \end{aligned}$$

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Converting regular expressions to FSA



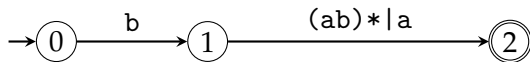
- For more complex expressions, one can replace the paths for individual symbols with corresponding automata
- Using ϵ transitions may ease the task
- The reverse conversion (from automata to regular expressions) is also easy:
 - identify the patterns on the left, collapse paths to single transitions with regular expressions

Exercise

convert $b((ab)^*|a)$ to an NFA

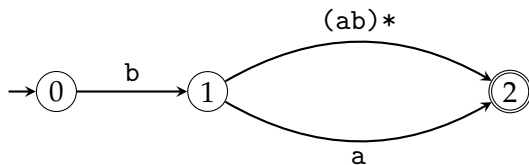
Exercise

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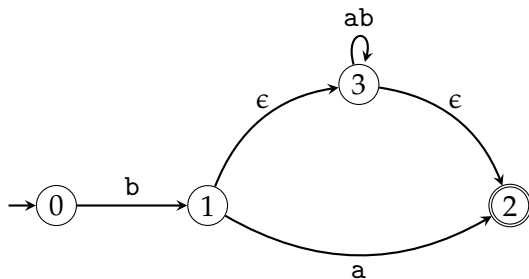
Exercise

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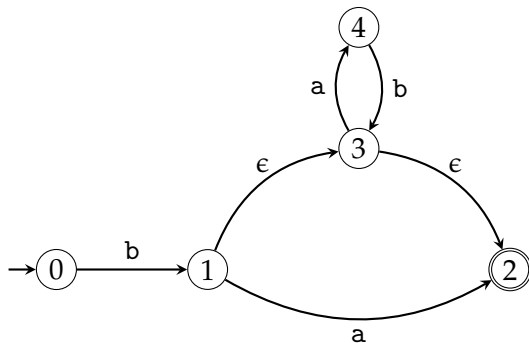
Exercise

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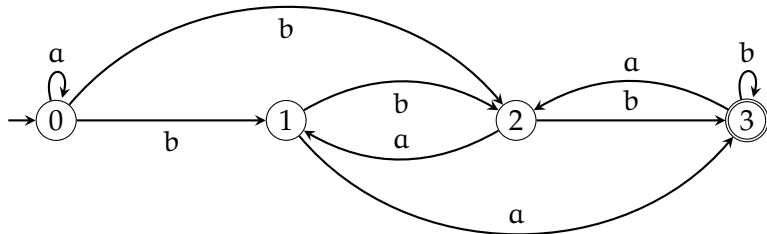


Exercise

convert $b((ab)^*|a)$ to an NFA

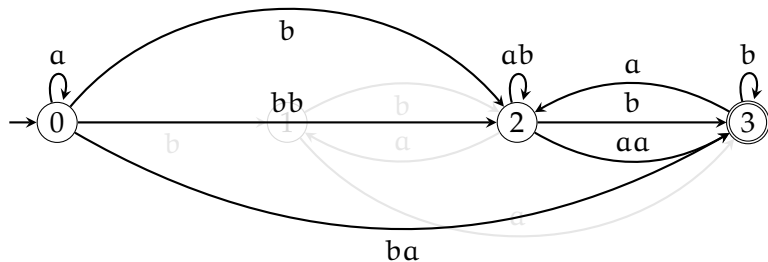


Converting FSA to regular expressions



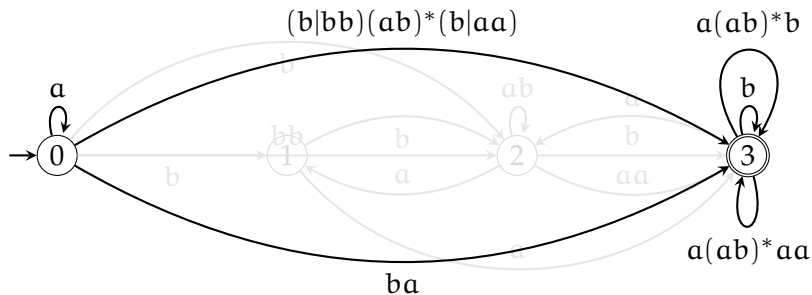
- The general idea: remove (intermediate) states, replacing edge labels with regular expressions

Converting FSA to regular expressions



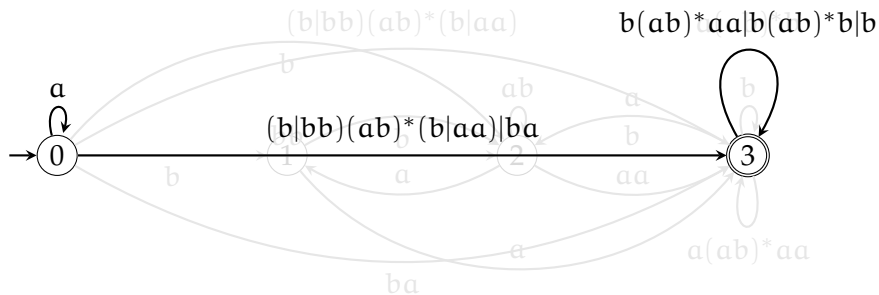
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Converting FSA to regular expressions



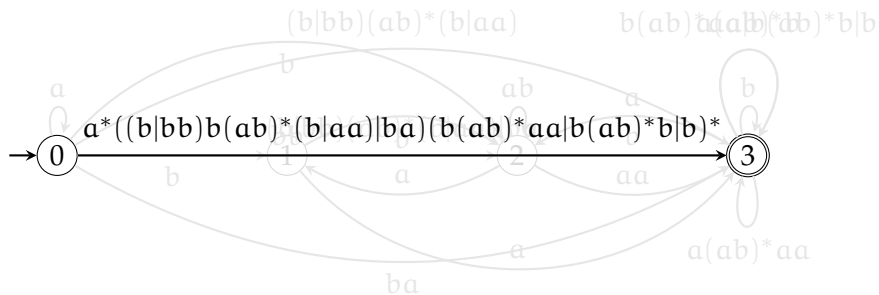
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Converting FSA to regular expressions



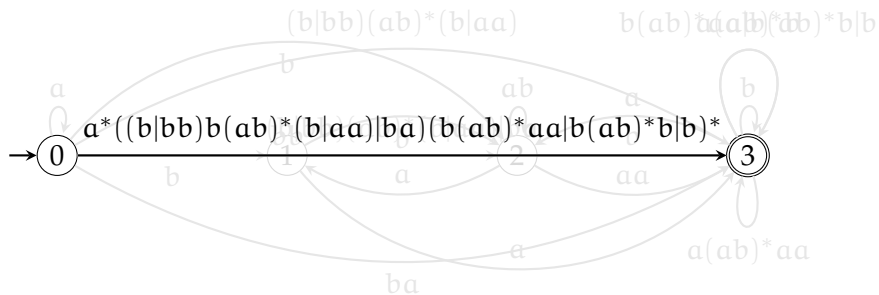
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Converting FSA to regular expressions



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Converting FSA to regular expressions



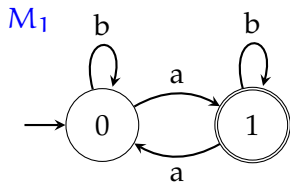
- The general idea: remove (intermediate) states, replacing edge labels with regular expressions

An exercise: simplify the resulting regular expressions

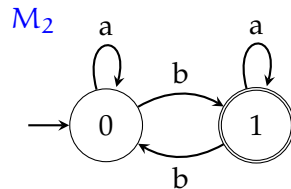
Two example FSA

what languages do they accept?

$$L_1 = \mathcal{L}(M_1)$$



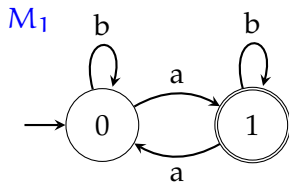
$$L_2 = \mathcal{L}(M_2)$$



Two example FSA

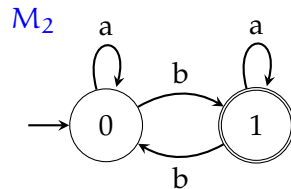
what languages do they accept?

$L_1 = \mathcal{L}(M_1)$



Odd number of a 's over $\{a, b\}$.

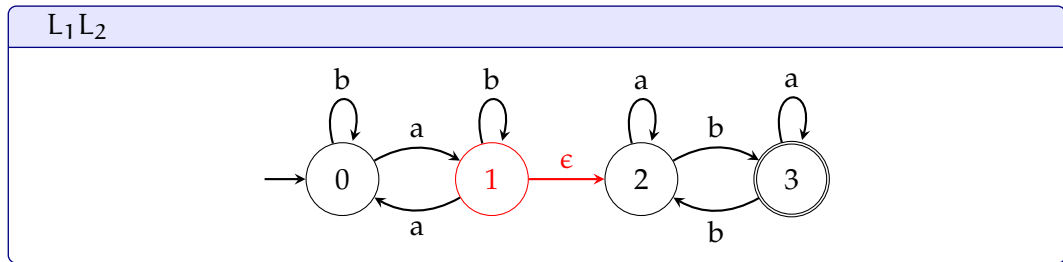
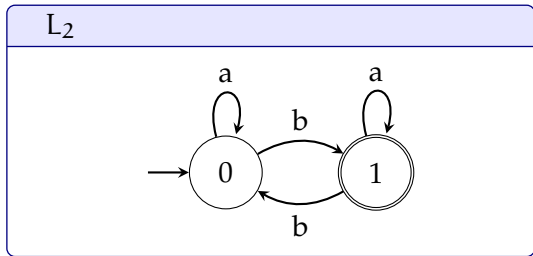
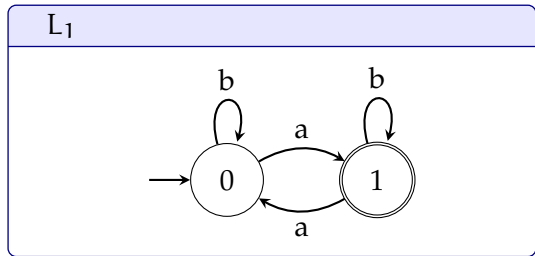
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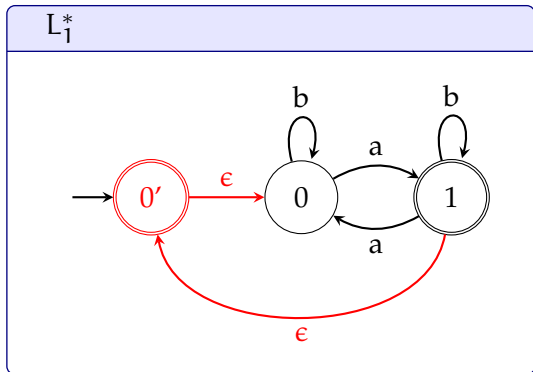
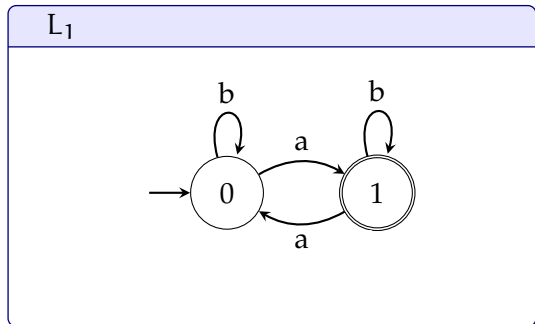
Odd number of b 's over $\{a, b\}$.

We will use these languages and automata for demonstration.

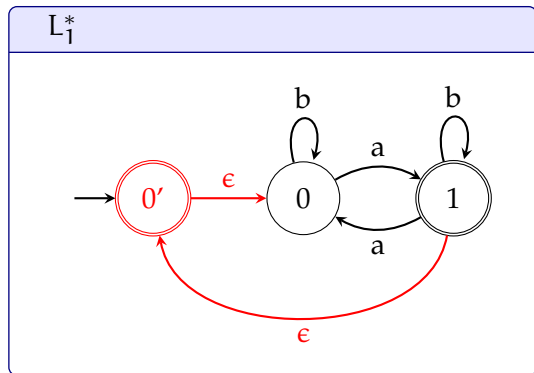
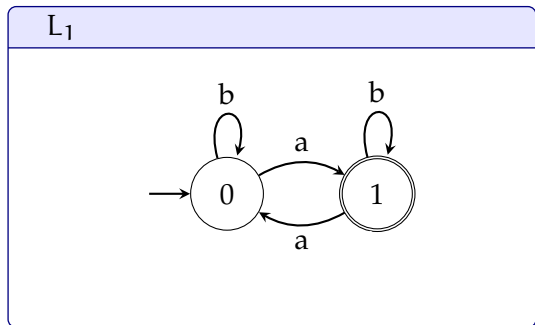
Concatenation



Kleene star

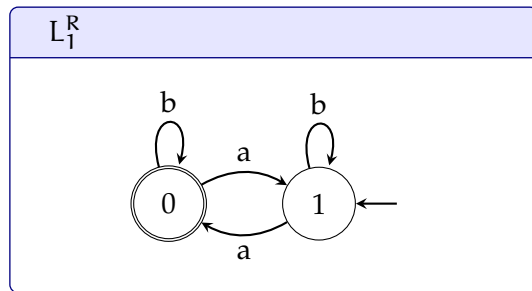
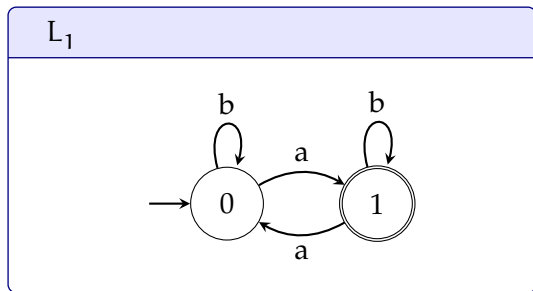


Kleene star

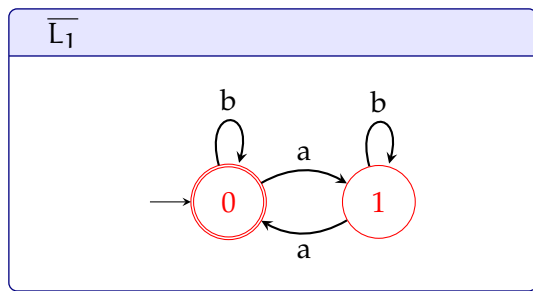
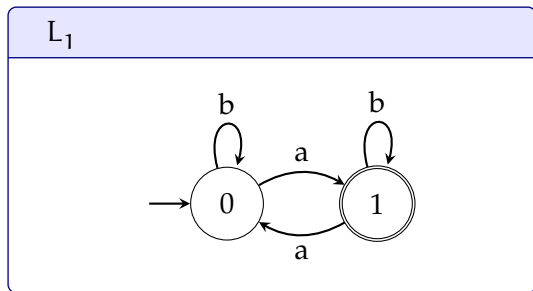


- What if there were more than one accepting states?

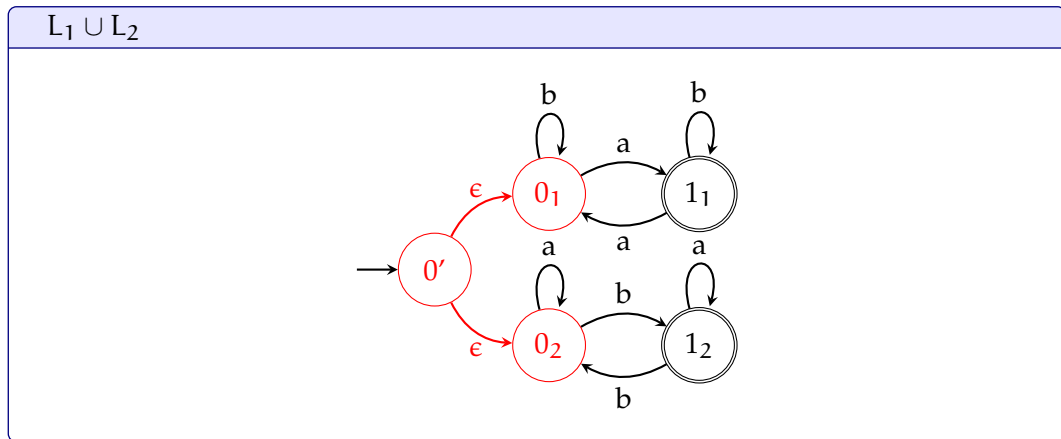
Reversal



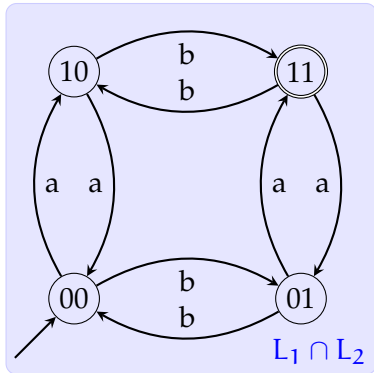
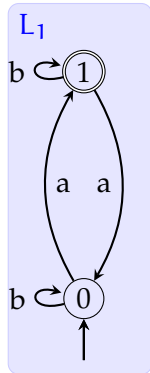
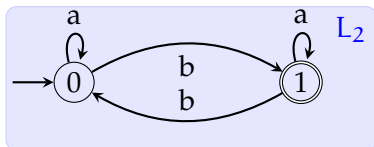
Complement



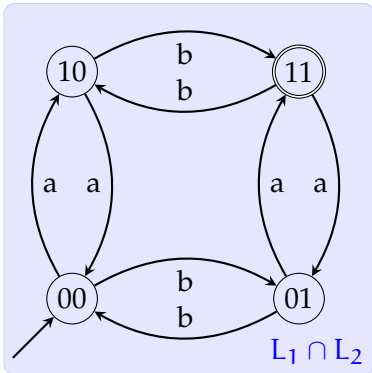
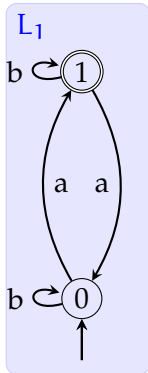
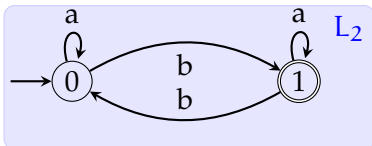
Union



Intersection



Intersection



...or

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

Closure properties of regular languages

- Since results of all the operations we studied are FSA: Regular languages are closed under
 - Concatenation
 - Kleene star
 - Reversal
 - Complement
 - Union
 - Intersection

Wrapping up

- FSA and regular expressions express regular languages
- Regular languages and FSA are closed under
 - Concatenation
 - Kleene star
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- To prove a language is regular, it is sufficient to find a regular expression or FSA for it
- To prove a language is not regular, we can use pumping lemma (see Appendix)

Wrapping up


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Next:

- Parsing

Acknowledgments, credits, references

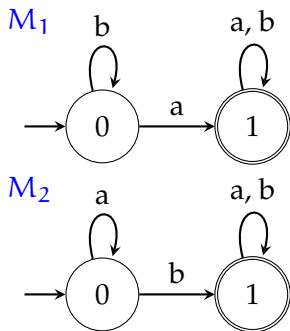
- The classic reference for FSA, regular languages and regular grammars is Hopcroft and Ullman (1979) (there are recent editions).

 Hopcroft, John E., Rajeev Motwani, and Jeffrey D. Ullman (2007). *Introduction to Automata Theory, Languages, and Computation*. 3rd. Pearson/Addison Wesley. ISBN: 9780321462251.

 Hopcroft, John E. and Jeffrey D. Ullman (1979). *Introduction to Automata Theory, Languages, and Computation*. Addison-Wesley Series in Computer Science and Information Processing. Addison-Wesley. ISBN: 9780201029888.

Another exercise on intersection

Construct the intersection of the automata below (adapted from Hopcroft, Motwani, and Ullman (2007), Fig. 4.4)



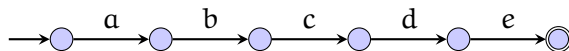
Is a language regular?

— or not

- To show that a language is regular, it is sufficient to find an FSA that recognizes it.
- Showing that a language is *not* regular is more involved
- We will study a method based on *pumping lemma*

Pumping lemma

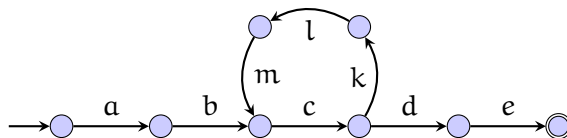
intuition



- What is the length of longest string generated by this FSA?

Pumping lemma

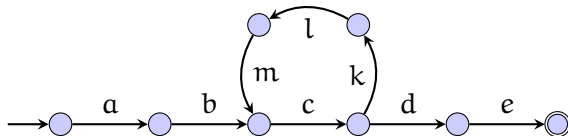
intuition



- What is the length of longest string generated by this FSA?

Pumping lemma

intuition



- What is the length of longest string generated by this FSA?
- Any FSA generating an infinite language has to have a loop (application of recursive rule(s) in the grammar)
- Part of every string longer than some number will include repetition of the same substring ('cklm' above)

Pumping lemma

definition

For every regular language L , there exist an integer p such that a string $x \in L$ can be factored as $x = uvw$,

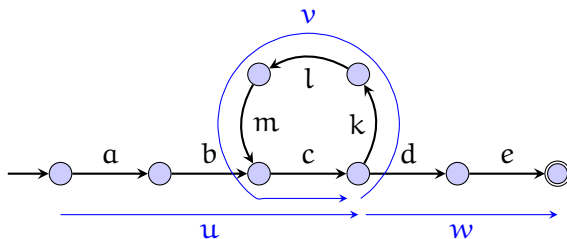
- $uv^i w \in L, \forall i \geq 0$
- $v \neq \epsilon$
- $|uv| \leq p$

Pumping lemma

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- $|uv| \leq p$



How to use pumping lemma

- We use pumping lemma to prove that a language is not regular
- Proof is by contradiction:
 - Assume the language is regular
 - Find a string x in the language, for all splits of $x = uvw$, at least one of the pumping lemma conditions does not hold
 - $uv^i w \in L$ ($\forall i \geq 0$)
 - $v \neq \epsilon$
 - $|uv| \leq p$

Pumping lemma example

prove $L = a^n b^n$ is not regular

- Assume L is regular: there must be a p such that, if uvw is in the language
 1. $uv^i w \in L$ ($\forall i \geq 0$)
 2. $v \neq \epsilon$
 3. $|uv| \leq p$
- Pick the string $a^p b^p$
- For the sake of example, assume $p = 5$, $x = aaaaaabbbbb$
- Three different ways to split

$\underbrace{a}_u \underbrace{aaa}_v \underbrace{abbbb}_w$	violates 1
$\underbrace{aaaa}_u \underbrace{ab}_v \underbrace{bbbb}_w$	violates 1 & 3
$\underbrace{aaaaab}_u \underbrace{bbb}_v \underbrace{b}_w$	violates 1 & 3
